

Structure of diffeomorphism-invariant Lagrangians on the product bundle of metrics and linear connections

J. Muñoz Masqué* and M. Eugenia Rosado María†

**Instituto de Física Aplicada, CSIC
C/ Serrano 144, 28006-Madrid, Spain email=jaime@iec.csic.es,*

†*Departamento de Matemática Aplicada
Escuela Técnica Superior de Arquitectura, UPM
Avda. Juan de Herrera 4, 28040-Madrid, Spain email=eugenia.rosado@upm.es*

Abstract. Let $p_C: C = CN \rightarrow N$ be the bundle of linear connections on a smooth manifold N and let $p_M: M \rightarrow N$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-) , $n^+ + n^- = n = \dim N$ on N . The structure of the first-order Lagrangians defined on the bundle $M \times_N C \rightarrow N$ that are invariant under the natural action of the diffeomorphisms of N , is determined.

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PRELIMINARIES

Jet bundles

Let $p: E \rightarrow N$ be an arbitrary fibred manifold, i.e., p is a surjective submersion; we set $\dim N = n$ and $\dim E = m + n$. An automorphism of p is a pair of diffeomorphisms $\phi: N \rightarrow N$, $\Phi: E \rightarrow E$ such that, $p \circ \Phi = \phi \circ p$. The group of all such automorphisms is denoted by $\text{Aut}(p)$, and its ‘Lie algebra’ is the space $\text{aut}(p) \subset \mathfrak{X}(E)$ of p -projectable vector fields on E . Equivalently, a vector field belongs to $\text{aut}(p)$ if and only if each transformation Φ_t of its flow belongs to $\text{Aut}(p)$. A natural group homomorphism $\text{Aut}(p) \rightarrow \text{Diff}N$, $\Phi \mapsto \phi$ exists, the kernel of which is the subgroup of vertical automorphisms of the fibred manifold, denoted by $\text{Aut}^v(p)$.

Latin (resp. Greek) indices run from 1 to n (resp. m). A system of coordinates (x^i, y^α) on an open subset $V \subseteq E$ is said to be a ‘fibred coordinate system’ for the submersion p if (x^i) is a coordinate system for N on $U = p(V)$.

Let $p^1: J^1E \rightarrow N$ be the 1-jet bundle of local sections of p , with natural projections $p^{1,0}: J^1E \rightarrow E$, $p^{1,0}(j_x^1s) = s(x)$, j_x^1s denoting the 1-jet at x of a section s of p defined on a neighbourhood of $x \in N$. A fibred coordinate system (x^i, y^α) on V induces a coordinate system $(x^i, y^\alpha, y_j^\alpha)$ on $(p^{1,0})^{-1}(V) = J^1V$ as follows: $y_{j,j}^\alpha(j_x^1s) = (\partial(y^\alpha \circ s)/\partial x^j)(x)$. Every morphism $\Phi: E \rightarrow E'$ whose associated map $\phi: N \rightarrow N'$ is a diffeomorphism, induces a map

$$\Phi^{(1)}: J^1E \rightarrow J^1E', \quad \Phi^{(1)}(j_x^1s) = j_{\phi(x)}^1(\Phi \circ s \circ \phi^{-1}). \quad (1)$$

If Φ_t is the flow of a vector field $X \in \text{aut}(p)$, then $\Phi_t^{(1)}$ is the flow of a vector field $X^{(1)} \in \mathfrak{X}(J^1 E)$, called the infinitesimal contact transformation of first order associated to the vector field X . The mapping $\text{aut}(p) \ni X \mapsto X^{(1)} \in \mathfrak{X}(J^1 E)$ is an injection of Lie algebras, namely, one has $(\lambda X + \mu Y)^{(1)} = \lambda X^{(1)} + \mu Y^{(1)}$, and $[X, Y]^{(1)} = [X^{(1)}, Y^{(1)}]$ for all $\lambda, \mu \in \mathbb{R}$, $X, Y \in \text{aut}(p)$. In particular,

$$\begin{aligned} X &= u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}, \quad u^i \in C^\infty(N), v^\alpha \in C^\infty(E), \\ X^{(1)} &= u^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha} + v_i^\alpha \frac{\partial}{\partial y_i^\alpha}, \quad v_i^\alpha = \frac{\partial v^\alpha}{\partial x^i} + y_i^\beta \frac{\partial v^\alpha}{\partial y^\beta} - y_k^\alpha \frac{\partial u^k}{\partial x^i}. \end{aligned}$$

The bundle $J^1(M(N) \times_N C(N))$

Let N be an n -dimensional orientable and oriented connected smooth manifold. Let $p_M: M = M(N) \rightarrow N$ (resp. $p_F: F(N) \rightarrow N$, resp. $p_C: C = C(N) \rightarrow N$) be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-) , $n^+ + n^- = n$ (resp. the bundle of linear frames, resp. linear connections) on N , see [3, 10]. Every coordinate system (x^i) on an open domain $U \subseteq N$ induces the following coordinate systems:

1. (x^i, y_{jk}) on $(p_M)^{-1}(U)$, where the functions $y_{jk} = y_{kj}$ are defined by,

$$g_x = \sum_{i \leq j} y_{ij}(g_x) (dx^i)_x \otimes (dx^j)_x, \quad \forall g_x \in (p_M)^{-1}(U).$$

2. (x^i, x_j^i) on $(p_F)^{-1}(U)$, where the functions x_j^i are defined by,

$$u = ((\partial/\partial x^1)_x, \dots, (\partial/\partial x^n)_x) \cdot (x_j^i(u)), \quad x = p_F(u), \forall u \in (p_F)^{-1}(U).$$

3. (x^i, A_{kl}^j) on $(p_C)^{-1}(U)$, where the functions A_{kl}^j are defined as follows. We first recall some basic facts. Connections on $F(N)$ (i.e., linear connections of N) are the splittings of the Atiyah sequence,

$$0 \rightarrow \text{ad}F(N) \rightarrow T_{Gl(n, \mathbb{R})}F(N) \xrightarrow{(p_F)_*} T(N) \rightarrow 0,$$

(cf. [1, 4, 7, 9]) where $\text{ad}F(N) = T^*(N) \otimes T(N)$ is the adjoint bundle, $T_{Gl(n, \mathbb{R})}(F(N)) = T(F(N))/Gl(n, \mathbb{R})$, see [3, 6], and $\text{gau}F(N) = \Gamma(N, \text{ad}F(N))$ is the gauge algebra of $F(N)$.

We think of $\text{gau}F(N)$ as the ‘Lie algebra’ of the gauge group $\text{Gau}F(N)$. Moreover, $p_C: C \rightarrow N$ is an affine bundle modelled over the vector bundle $\otimes^2 T^*(N) \otimes T(N)$. The section of p_C induced tautologically by the linear connection Γ is denoted by $s_\Gamma: N \rightarrow C$. Every $B \in \mathfrak{gl}(n, \mathbb{R})$ defines a one-parameter group $\varphi_t^B: U \times Gl(n, \mathbb{R}) \rightarrow U \times Gl(n, \mathbb{R})$ of gauge transformations by setting (cf. [3]), $\varphi_t^B(x, \Lambda) = (x, \exp(tB) \cdot \Lambda)$. Let us denote by $\tilde{B} \in \text{gau}(p_F)^{-1}(U)$ the corresponding infinitesimal generator. If (E_j^i) is

the standard basis of $\mathfrak{gl}(n, \mathbb{R})$, then $\tilde{E}_j^i = \sum_{h=1}^n x_h^j \partial / \partial x_h^i$, is a basis of $\text{gau}(p_F)^{-1}(U)$. Let $\tilde{E}_j = \tilde{E}_j^i \text{mod} G$ be the class of \tilde{E}_j^i on $\text{ad}F(N)$. Unique smooth functions A_{jk}^i on $(p_C)^{-1}(U)$ exist such that,

$$s_\Gamma \left(\frac{\partial}{\partial x^j} \right) = \frac{\partial}{\partial x^j} - (A_{jk}^i \circ \Gamma) \tilde{E}_j^i$$

for every s_Γ and $A_{jk}^i(\Gamma_x) = \Gamma_{jk}^i(x)$, where Γ_{jk}^i are the Christoffel symbols of the linear connection Γ in the coordinate system (x^i) , see [8, III, Proposition 7.4].

Natural lifts

Let $f_M: M \rightarrow M$, cf. [10] (resp. $\tilde{f}: F(N) \rightarrow F(N)$, cf. [8, p. 226]) be the natural lift of $f \in \text{Diff}N$ to the bundle of metrics (resp. linear frame bundle); namely $f_M(g_x) = (f^{-1})^* g_x$ (resp. $\tilde{f}(X_1, \dots, X_n) = (f_* X_1, \dots, f_* X_n)$, where $(X_1, \dots, X_n) \in F_x(N)$); hence $p_M \circ f_M = f \circ p_M$ (resp. $p_F \circ \tilde{f} = f \circ p_F$), and $f_M: M \rightarrow M$ (resp. $\tilde{f}: F(N) \rightarrow F(N)$) have a natural extension to 1-jet bundles $f_M^{(1)}: J^1(M) \rightarrow J^1(M)$ (resp. $\tilde{f}^{(1)}: J^1(FN) \rightarrow J^1(FN)$) as defined in the formula (1), i.e.,

$$f_M^{(1)}(j_x^1 g) = j_{f(x)}^1 (f_M \circ g \circ f^{-1}) \quad (\text{resp. } \tilde{f}^{(1)}(j_x^1 s) = j_{\tilde{f}(x)}^1 (\tilde{f} \circ s \circ f^{-1})).$$

As \tilde{f} is an automorphism of the principal $GL(n, \mathbb{R})$ -bundle $F(N)$, it acts on linear connections by pulling back connection forms, i.e., $\Gamma' = \tilde{f}^*(\Gamma)$ where $\omega_{\Gamma'} = (\tilde{f}^{-1})^* \omega_\Gamma$ (see [8, II, Proposition 6.2-(b)], [3, 3.3]). Hence, a unique diffeomorphism $\tilde{f}_C: C \rightarrow C$ exists such that, for every linear connection Γ , 1) $p_C \circ \tilde{f}_C = f \circ p_C$ and 2) $\tilde{f}_C \circ s_\Gamma = s_{\tilde{f}(\Gamma)}$.

If f_t is the flow of a vector field $X \in \mathfrak{X}(N)$, then the infinitesimal generator of $(f_t)_M$ (resp. \tilde{f}_t , resp. $(\tilde{f}_t)_C$) in $\text{Diff}M$ (resp. $\text{Diff}F(N)$, resp. $\text{Diff}C$) is denoted by X_M (resp. \tilde{X} , resp. \tilde{X}_C) and the following Lie-algebra homomorphisms are obtained:

$$\begin{cases} \mathfrak{X}(N) \rightarrow \mathfrak{X}(M), & X \mapsto X_M, \\ \mathfrak{X}(N) \rightarrow \mathfrak{X}(F(N)), & X \mapsto \tilde{X}, \\ \mathfrak{X}(N) \rightarrow \mathfrak{X}(C), & X \mapsto \tilde{X}_C. \end{cases}$$

If $X = u^i \partial / \partial x^i \in \mathfrak{X}(N)$ is the local expression for X , then

1. From ([10, eqs. (2)-(4)]) we know that the natural lift of X to M is given by,

$$X_M = u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{ih} \right) \frac{\partial}{\partial y_{ij}} \in \mathfrak{X}(M).$$

and its 1-jet prolongation,

$$\begin{aligned} X_M^{(1)} = & u^i \frac{\partial}{\partial x^i} - \sum_{i \leq j} \left(\frac{\partial u^h}{\partial x^i} y_{hj} + \frac{\partial u^h}{\partial x^j} y_{hi} \right) \frac{\partial}{\partial y_{ij}} \\ & - \sum_{i \leq j} \left(\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} + \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{hi} \right. \\ & \left. + \frac{\partial u^h}{\partial x^i} y_{hj,k} + \frac{\partial u^h}{\partial x^j} y_{hi,k} + \frac{\partial u^h}{\partial x^k} y_{ij,h} \right) \frac{\partial}{\partial y_{ij,k}}. \end{aligned}$$

2. From [5, Proposition 3](also see [8, VI, Proposition 21.1]) we know that the natural lift of X to $F(N)$ is given by,

$$\tilde{X} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x_j^l \frac{\partial}{\partial x_j^i},$$

and its 1-jet prolongation,

$$\tilde{X}^{(1)} = u^i \frac{\partial}{\partial x^i} + \frac{\partial u^i}{\partial x^l} x_j^l \frac{\partial}{\partial x_j^i} + \left(\frac{\partial u^i}{\partial x^r} x_{j,k}^r - \frac{\partial u^r}{\partial x^k} x_{j,r}^i + \frac{\partial^2 u^i}{\partial x^k \partial x^r} x_j^r \right) \frac{\partial}{\partial x_{j,k}^i}.$$

3. Finally,

$$\tilde{X}_C = u^i \frac{\partial}{\partial x^i} - \left(\frac{\partial^2 u^i}{\partial x^k \partial x^j} - \frac{\partial u^i}{\partial x^r} A_{jk}^r + \frac{\partial u^s}{\partial x^k} A_{js}^i + \frac{\partial u^l}{\partial x^j} A_{lk}^i \right) \frac{\partial}{\partial A_{jk}^i},$$

$$\begin{aligned} \tilde{X}_C^{(1)} = & u^i \frac{\partial}{\partial x^i} + w_{jk}^i \frac{\partial}{\partial A_{jk}^i} + w_{jkh}^i \frac{\partial}{\partial A_{jk,h}^i}, \\ w_{jk}^i = & -\frac{\partial^2 u^i}{\partial x^k \partial x^j} + \frac{\partial u^i}{\partial x^r} A_{jk}^r - \frac{\partial u^s}{\partial x^k} A_{js}^i - \frac{\partial u^r}{\partial x^j} A_{rk}^i, \end{aligned} \quad (2)$$

$$\begin{aligned} w_{jkh}^i = & -\frac{\partial^3 u^i}{\partial x^h \partial x^k \partial x^j} + \frac{\partial^2 u^i}{\partial x^h \partial x^r} A_{jk}^r - \frac{\partial^2 u^s}{\partial x^h \partial x^k} A_{js}^i - \frac{\partial^2 u^r}{\partial x^h \partial x^j} A_{rk}^i \\ & + \frac{\partial u^i}{\partial x^r} A_{jk,h}^r - \frac{\partial u^s}{\partial x^k} A_{js,h}^i - \frac{\partial u^l}{\partial x^j} A_{lk,h}^i - \frac{\partial u^t}{\partial x^h} A_{jk,t}^i. \end{aligned} \quad (3)$$

Let $p: M \times_N C \rightarrow N$ be the natural projection. We denote by $\tilde{f} = (f_M, f_C)$ (resp. $\tilde{X} = (X_M, \tilde{X}_C) \in \mathfrak{X}(M \times_N C)$) the natural lift of f (resp. X) to $M \times_N C$. The prolongation to the bundle $J^1(M \times_N C)$ of \tilde{X} is as follows:

$$\tilde{X}^{(1)} = (X_M^{(1)}, \tilde{X}_C^{(1)}) = u^i \frac{\partial}{\partial x^i} + \sum_{i \leq j} v_{ij} \frac{\partial}{\partial y_{ij}} + \sum_{i \leq j} v_{ijk} \frac{\partial}{\partial y_{ij,k}} + w_{jk}^i \frac{\partial}{\partial A_{jk}^i} + w_{jkh}^i \frac{\partial}{\partial A_{jk,h}^i},$$

where

$$\begin{aligned} v_{ij} = & -\frac{\partial u^h}{\partial x^i} y_{hj} - \frac{\partial u^h}{\partial x^j} y_{hi}, \\ v_{ijk} = & -\frac{\partial^2 u^h}{\partial x^i \partial x^k} y_{hj} - \frac{\partial^2 u^h}{\partial x^j \partial x^k} y_{ih} - \frac{\partial u^h}{\partial x^i} y_{hj,k} - \frac{\partial u^h}{\partial x^j} y_{ih,k} - \frac{\partial u^h}{\partial x^k} y_{ij,h}, \end{aligned}$$

and w_{jk}^i, w_{jkh}^i are given in the formulas (2), (3), respectively.

Diff N - and $\mathfrak{X}(N)$ -invariance

A differential form $\omega_r \in \Omega^r(J^1(M \times_N C))$, $r \in \mathbb{N}$, is said to be Diff N -invariant—or invariant under diffeomorphisms—(resp. $\mathfrak{X}(N)$ -invariant) if the following equation holds: $(\bar{f}^{(1)})^* \omega_r = \omega_r$, $\forall f \in \text{Diff}N$ (resp. $L_{\bar{X}(1)} \omega_r = 0$, $\forall X \in \mathfrak{X}(N)$). Obviously, “Diff N -invariance” implies “ $\mathfrak{X}(N)$ -invariance” and the converse is almost true. Because of this, below we consider $\mathfrak{X}(N)$ -invariance only.

A linear frame $(X_1, \dots, X_n) \in F_x(N)$ is said to be orthonormal with respect to $g_x \in M_x(N)$ (or simply g_x -orthonormal) if $g_x(X_i, X_j) = 0$ for $1 \leq i < j \leq n$, $g(X_i, X_i) = 1$ for $1 \leq i \leq n^+$, $g(X_i, X_i) = -1$ for $n^+ + 1 \leq i \leq n$.

As N is an oriented manifold, there exists a unique p -horizontal n -form \mathbf{v} on $M \times_N C$ such that, $\mathbf{v}_{(g_x, \Gamma_x)}(X_1, \dots, X_n) = 1$, for every g_x -orthonormal basis (X_1, \dots, X_n) belonging to the orientation of N . Locally $\mathbf{v} = \rho v_n$, where $\rho = \sqrt{(-1)^{n^-} \det(y_{ij})}$ and $v_n = dx^1 \wedge \dots \wedge dx^n$. As proved in [10, Proposition 7], the form \mathbf{v} is Diff N -invariant and hence $\mathfrak{X}(N)$ -invariant. A Lagrangian density Λ on $J^1(M \times_N C)$ can be globally written as $\Lambda = \mathcal{L} \mathbf{v}$ for a unique function $\mathcal{L} \in C^\infty(J^1(M \times_N C))$ and Λ is $\mathfrak{X}(N)$ -invariant if and only if the function \mathcal{L} is $\mathfrak{X}(N)$ -invariant; that is $\bar{X}^{(1)}(\mathcal{L}) = 0$, $\forall X \in \mathfrak{X}(N)$. Therefore, the invariance of Lagrangian densities is reduced to that of scalar functions.

As the values for u^h , $\partial u^h / \partial x^i$, $\partial^2 u^h / \partial x^i \partial x^j$ ($i \leq j$), and $\partial^3 u^h / \partial x^i \partial x^j \partial x^k$ ($i \leq j \leq k$) at a point $x \in N$ can be taken arbitrarily, we deduce that the equation $\bar{X}^{(1)}(\mathcal{L}) = 0$, $\forall X \in \mathfrak{X}(N)$ is equivalent to the following system of partial differential equations

$$\begin{aligned} 0 &= \frac{\partial}{\partial x^i}(\mathcal{L}), & \forall i, \\ 0 &= X_h^i(\mathcal{L}), & \forall h, i, \\ 0 &= X_h^{ik}(\mathcal{L}), & \forall h, i \leq k, \\ 0 &= X_i^{jkh}(\mathcal{L}), & \forall i, j \leq k \leq h, \end{aligned} \quad (4)$$

where

$$\begin{aligned} X_h^i &= -y_{hi} \frac{\partial}{\partial y_{ii}} - y_{hj} \frac{\partial}{\partial y_{ij}} - y_{ih,k} \frac{\partial}{\partial y_{ii,k}} - y_{hj,k} \frac{\partial}{\partial y_{ij,k}} - \sum_{s \leq j} y_{sj,h} \frac{\partial}{\partial y_{sj,i}} + A_{jk}^i \frac{\partial}{\partial A_{jk}^h} \\ &\quad - A_{jh}^r \frac{\partial}{\partial A_{ji}^r} - A_{hk}^r \frac{\partial}{\partial A_{ik}^r} + A_{jk,s}^i \frac{\partial}{\partial A_{jk,s}^h} - A_{jh,r}^s \frac{\partial}{\partial A_{ji,r}^s} - A_{hk,r}^s \frac{\partial}{\partial A_{ik,r}^s} - A_{jk,h}^r \frac{\partial}{\partial A_{jk,i}^r}, \\ X_h^{ik} &= -y_{ih} \frac{\partial}{\partial y_{ii,k}} - y_{kh} \frac{\partial}{\partial y_{kk,i}} - y_{hj} \frac{\partial}{\partial y_{ij,k}} - y_{hj} \frac{\partial}{\partial y_{kj,i}} - \frac{\partial}{\partial A_{ik}^h} - \frac{\partial}{\partial A_{ki}^h} \\ &\quad + A_{js}^k \frac{\partial}{\partial A_{js,i}^h} - A_{jh}^s \frac{\partial}{\partial A_{jk,i}^s} - A_{hr}^s \frac{\partial}{\partial A_{kr,i}^s} + A_{js}^i \frac{\partial}{\partial A_{js,k}^h} - A_{jh}^s \frac{\partial}{\partial A_{ji,k}^s} - A_{hr}^s \frac{\partial}{\partial A_{ir,k}^s}, \\ X_i^{jkh} &= \frac{\partial}{\partial A_{jk,h}^i} + \frac{\partial}{\partial A_{jh,k}^i} + \frac{\partial}{\partial A_{hk,j}^i} + \frac{\partial}{\partial A_{hj,k}^i} + \frac{\partial}{\partial A_{kj,h}^i} + \frac{\partial}{\partial A_{kh,j}^i}, \end{aligned}$$

From the above expressions it is easy to prove that the number of equations in system (4) is

$$n + n^2 + n \binom{n+1}{2} + n \binom{n+2}{3} = n \binom{n+3}{3}.$$

As a simple computation shows, we have

Proposition 1 *The vector fields $\partial/\partial x^i, X_h^i, X_h^{ik}, X_i^{jkh}$ are linearly independent and span an involutive distribution on $J^1(M \times_N C)$ of rank $n \binom{n+3}{3}$. Hence, the number of functionally invariant Lagrangians on $J^1(M \times_N C)$ is $\frac{1}{6}(5n^4 + 3n^3 - 5n^2 + 3n)$.*

THE FACTORIZATION RESULT FOR THE INVARIANTS

Let B_N be the sub-vector bundle of all the tensors $t \in \wedge^2 T^*N \otimes T^*N \otimes TN$ such that $\mathfrak{S}_{X,Y,Z}(X,Y,Z) = 0$, $\forall X,Y,Z \in T_x N$, $\forall x \in N$, and let consider the following mapping

$$\Upsilon: J^1(M \times_N C) \rightarrow M \times_N \left((\wedge^2 T^*N \otimes TN) \oplus (\wedge^2 T^*N \otimes T^*N \otimes TN) \oplus B_N \right),$$

$$\Upsilon(j_x^1(g, s_\Gamma)) = (g_x, (\nabla^\Gamma - \nabla^g)_x, (R^\Gamma)_x, \Upsilon^3(j_x^1(g, s_\Gamma))),$$

where $\Upsilon^3(j_x^1(g, s_\Gamma))(X,Y,Z) = (\nabla_Z^\Gamma T^\Gamma)(X,Y) + T^\Gamma(T^\Gamma(X,Y),Z) - R^\Gamma(X,Y)(Z)$.

Theorem 2 *Every $\mathfrak{X}(N)$ -invariant Lagrangian function \mathcal{L} on $J^1(M \times_N C)$ factors through Υ as follows: $\mathcal{L} = \tilde{\mathcal{L}} \circ \Upsilon$, where*

$$\tilde{\mathcal{L}}: M \times_N \left((\otimes^2 T^*N \otimes TN) \oplus (\wedge^2 T^*N \otimes T^*N \otimes TN) \oplus B_N \right) \rightarrow \mathbb{R}$$

*is a smooth function that is, in turn, invariant under the natural action of $\mathfrak{X}(N)$ on such tensorial bundle, namely, $\hat{X}(\tilde{\mathcal{L}}) = 0$, for all $X \in \mathfrak{X}(N)$, where \hat{X} denotes the natural lift of $X \in \mathfrak{X}(N)$ to $M \times_N \left((\otimes^2 T^*N \otimes TN) \oplus (\wedge^2 T^*N \otimes T^*N \otimes TN) \oplus B_N \right)$.*

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